Parametric Completeness for Separation Theories

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Joint work with James Brotherston (UCL)

Logics: Expressivity vs Complexity

Mathematical logics expressivity trade-off

- Weaker languages cannot capture interesting properties, but
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This talk

- Study this gap in the context of separation logic

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Models of Separation Logic and BBI

- Models of BBI: partial commutative relational monoids
- Concrete model: Heaps : Location $\rightharpoonup$ Values
- In-between: **separation theories** satisfying some of
  functionality cancellativity single-unit ...

Definability of Classes of Models

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4. We show how to axiomatise validity in our hybrid system(s). Moreover, we do this such that completeness is **parametric** in the axioms defining separation theories.

Boolean BI

(Propositional) Boolean BI

BBI formula

$$A ::= P \mid \top \mid \bot \mid \neg A \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid A_1 \rightarrow A_2$$
$$\mid \mathrm{I} \mid A_1 * A_2 \mid A_1 \mathrel{-\!\!*} A_2$$

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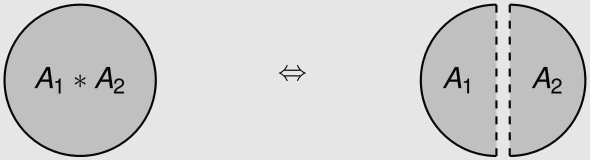
Separating Conjunction $*$



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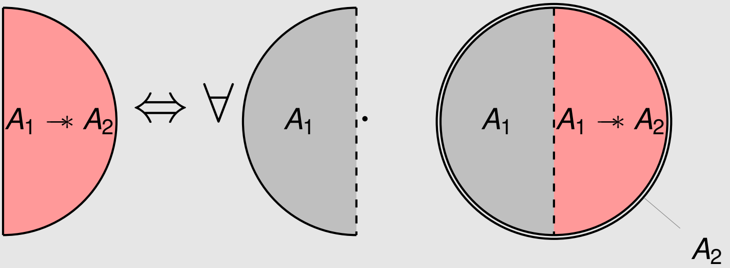
Magic Wand $\mathrel{-\!\!*}$



Proof theory of BBI

Provability for the multiplicatives is given by

$$A * B \vdash B * A \qquad A * (B * C) \vdash (A * B) * C$$

$$A \vdash A * \mathrm{I} \qquad A * \mathrm{I} \vdash A$$

$$\frac{A_1 \vdash B_1 \quad A_2 \vdash B_2}{A_1 * A_2 \vdash B_1 * B_2} \qquad \frac{A * B \vdash C}{A \vdash B \mathrel{-\!\!*} C} \qquad \frac{A \vdash B \mathrel{-\!\!*} C}{A * B \vdash C}$$

BBI-models

BBI model $\langle W, \circ, E \rangle$

A **relational commutative monoid**, *i.e* a tuple $\langle W, \circ, E \rangle$ where

- $\circ : W \times W \to \mathcal{P}(W)$

$$\left(\text{lifted to } W_1 \circ W_2 \stackrel{\text{def}}{=} \bigcup_{w_1 \in W_1, w_2 \in W_2} w_1 \circ w_2\right)$$

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- e is the empty heap that is undefined everywhere.

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Forcing relation $M, w \models_\rho A$ $\quad M = \langle W, \circ, E \rangle$

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Theorem — Galmiche and Larchey-Wendling 2006

Provability in BBI *coincides with validity in* BBI*-models.*

(Un)definable properties in BBI

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Cross-split property: whenever $(a \circ b) \cap (c \circ d) \neq \emptyset$, there exist ac, ad, bc, bd such that $a \in ac \circ ad$, $b \in bc \circ bd$, $c \in ac \circ bc$ and $d \in ad \circ bd$.

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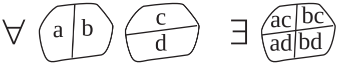

Separation Algebras throughout the Ages

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A class $\mathcal{C}$ of BBI-models is said to be **$\mathcal{L}$-definable** if there exists an $\mathcal{L}$-formula A such that for all BBI-models M,

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Divisibility:	$\neg\mathrm{I} \vdash \neg\mathrm{I} * \neg\mathrm{I}$

Proof.

Just directly verify the needed biimplication. □

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Proof.

Structural induction on A. □

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Lemma

Let $\mathcal{C}$ *be a class of* BBI*-models, and suppose that there exist* BBI*-models* M_1 *and* M_2 *such that* $M_1, M_2 \in \mathcal{C}$ *but* $M_1 \uplus M_2 \notin \mathcal{C}$*. Then* $\mathcal{C}$ *is not* BBI*-definable.*

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The disjoint union of any two single-unit BBI-models (e.g. two copies of $\mathbb{N}$ under addition) is not a single-unit model, so we are done by the above Lemma. □

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Proof.

E.g., for functionality, we build models M and M' such that there is a bounded morphism from M to M', but M is functional while M' is not. See paper for details. □

Hybrid BBI

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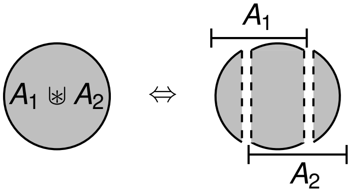
Proof.

Easy verifications! □

Overlapping conjunction



$$M, w \models_\rho A_1 \circledast A_2 \Leftrightarrow \exists w_1, w_2, w_3, w', w'' \in W.$$
$$w' \in w_1 \circ w_2 \text{ and } w'' \in w_2 \circ w_3 \text{ and } w \in w' \circ w_3$$
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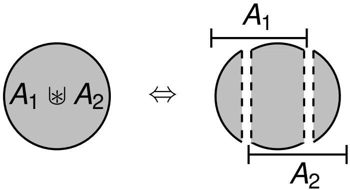

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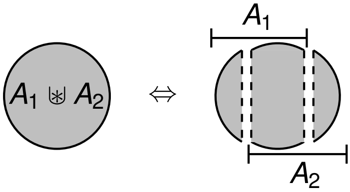

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(but where does ℓ_s come from?..)

A word about cross-split

We have brushed over the **cross-split** property:

$(a \circ b) \cap (c \circ d) \neq \emptyset$, *implies* $\exists ac, ad, bc, bd$ *with* $a \in ac \circ ad$, $b \in bc \circ bd$, $c \in ac \circ bc$, $d \in ad \circ bd$.

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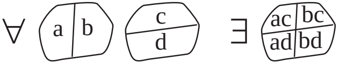

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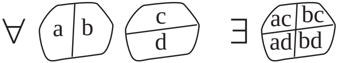
> 
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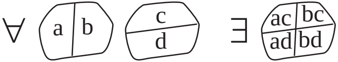

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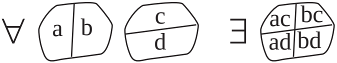

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then cross-split is definable as the pure formula

$$\begin{aligned}(a * b) \wedge (c * d) \vdash @_a(\top * \downarrow ac.\, @_a(\top * \downarrow ad.\, @_a(ac * ad) \\ \wedge @_b(\top * \downarrow bc.\, @_b(\top * \downarrow bd.\, @_b(bc * bd) \\ \wedge @_c(ac * bc) \wedge @_d(ad * bd)))))\end{aligned}$$

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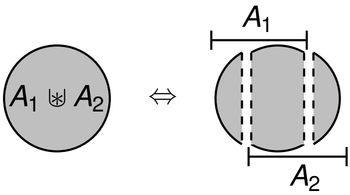

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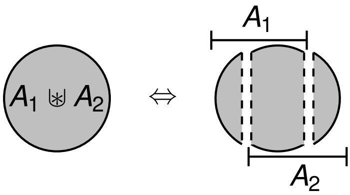

Proposition

$A_1 \circledast A_2$ is definable via the following HyBBI(↓) *formula, where ℓ and ℓ_s do not occur in A_1 or A_2:*

$$\downarrow\ell.\, \top * \downarrow\ell_s.\, @_\ell(\ell_s \multimap\!\circledast\, A_1) * (\ell_s \multimap\!\circledast\, A_2) * \ell_s$$

(where $A \multimap\!\circledast\, B \stackrel{\text{def}}{=} \neg(A \multimap\!\!* \neg B)$)

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Proposition — **Soundness**

Any $\mathbf{K}_{\mathrm{HyBBI}(\downarrow)}$*-provable sequent is valid in all* BBI*-models.*

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(In our case, we also have to show that the canonical model is really a BBI-model.)

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In particular, we obtain sound and complete proof systems for separation algebras.

Conclusion

Conclusions and future work

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Thanks for listening!

Draft paper available from authors' webpages:

J. Brotherston and J. Villard.
Parametric completeness for separation theories.
To be presented at POPL'14.

Parametric Completeness for Separation Theories

Jules Villard

University College London
Programming Principles, Logic and Verification Group

Joint work with James Brotherston (UCL)